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Laminar natural convection boundary-layer flow along a heated vertical plate in a stratified environment

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Abstract—All similarity solutions of the laminar natural convection boundary-layer equations for air are numerically determined for a fixed wall and variable environment temperature. It is found that the positive M class does not have the singularity found by Merkin for a variable wall and fixed environment temperature. Solutions of the negative M class for an unstable stratification depend on the position of the outer edge and are unusable. The similarity solutions for a stable stratification show regions of backflow. Therefore, the calculation of non-similar solutions of the boundary-layer equations along a heated vertical plate with a sharp leading edge requires that the solution is known at the end of the plate. The positive M class provides such a solution for a semi-infinite plate. If the environment temperature becomes equal to the wall temperature at a finite distance x_0, the non-similar solution does not smoothly approach the negative M class similarity solution close to x_0.

1. INTRODUCTION

If the characteristic number of the natural convection flow, i.e. the Grashof number, is increased to infinity, the Navier–Stokes flow along a heated vertical plate becomes identical to the solution of the boundary-layer equations. If the environment is isothermal and stagnant, a coordinate transformation exists which simplifies the boundary-layer equations to ordinary differential equations. The solution of this system, which is a similarity solution of the boundary-layer equations, was numerically determined by Ostrach [1]. If the vertical plate is part of an enclosure, the environment of the plate (the core of the enclosure) will not be isothermal, but stratified. This stratification will be stable: the temperature increases with height. An unstable stratification is only of theoretical interest; it is a solution of the steady Navier–Stokes equations, but unsteady effects will cause a transition to a steady solution in which the stratification is broken up.

Semenov [2] derived the system of ordinary differential equations for all possible distributions of the wall and environment temperature leading to a similarity solution of the boundary-layer equations. Some solutions of Semenov's system are already known in the literature; Ostrach [1], giving the solution for a fixed wall and environment temperature, Sparrow and Gregg [3], giving part of the class with variable wall and fixed environment temperature, and Cheesewright [4] and Yang *et al.* [5], giving part of the class with fixed wall and variable environment temperature. Recently Merkin [6] found that the similarity solution for a variable wall and fixed environment temperature becomes singular if a critical value of the parameter describing the wall temperature is exceeded. It is investigated here whether such a singular behaviour is also found for a fixed wall and variable environment temperature. Further, the differential equations for the new class of similarity solutions are numerically solved.

Once the similarity solutions have been determined, it has to be investigated in which part of the boundary layer they hold. Firstly, the solution must be matchable with the environment solution: the velocity and temperature profiles have to be independent of the position of the far outer edge of the boundary layer. Secondly, a similarity solution holds for small x (coordinate along the plate), if it is matchable to the solution in a small region, $O(Gr^{-1/2})$, at the leading edge of the plate, where boundary-layer equations do not apply, but Navier–Stokes equations have to be used. If this is not the case, the similarity solution found might be the boundary-layer solution for large x. To check this we also solved the full (non-similar) boundary-layer equations.

When this paper was in preparation, a related paper by Kulkarni *et al.* [7] was published. They determined a similarity solution for a fixed wall temperature and a linear, stably stratified environment. The authors claimed to have found a new class of similarity solutions, but this class was already detected by Semenov [2]. Actually Semenov's new class is more general, because the parameter describing the variation of the environment temperature can be any real number, whereas it has to be an integer in the description of Kulkarni *et al.* The present paper determines the solutions of the new class for the whole range of

NOMENCLATURE

$f(\eta)$	similarity stream function
$F(\bar{\eta})$	similarity stream function (large n formulation)
g	gravitational acceleration
$g(\eta)$	similarity temperature
$G(\bar{\eta})$	similarity temperature (large n formulation)
Gr	Grashof number, $g\beta\Delta T x_0^3/\nu^2$
h	f'
j	gridpoint numbering in the η-direction
m	parameter describing whether the environment temperature ($m = 0$) or the wall temperature ($m = -1$) is fixed
M	coefficient in the ξ-coordinate
n	parameter describing the variation of the wall and/or environment temperature
N	coefficient in the ξ-coordinate
Nu	Nusselt number, $-x_0[\partial(T-T_\infty(0))/\partial y]_w/\Delta T$
p	pressure
Pr	Prandtl number
s	iterative level in the numerical procedure
T	temperature
T_c	constant temperature
ΔT	characteristic temperature difference, $T_w - T_\infty(0)$
u	vertical velocity component
v	velocity component perpendicular to the plate
x	vertical coordinate
x_0	length scale
y	coordinate perpendicular to, and beginning at, the plate.

Greek symbols

β	coefficient of thermal expansion
η	similarity y-coordinate
$\bar{\eta}$	similarity y-coordinate (large n formulation, $\bar{\eta} = \|n\|^{1/4}\eta$)
$\Delta\eta$	gridsize in the η-direction
ν	molecular kinematic viscosity
ξ	transformed x-coordinate, $Mx+N$
ρ	density
ψ	stream function.

Superscript

$'$	differentiation with respect to η (or $\bar{\eta}$).

Subscripts

w	wall condition
∞	environment condition.

parameters describing the variation of the stratification. By comparison with a non-similar boundary-layer calculation we will show that the new similarity solution for the stable, linear stratification does not fit in the boundary-layer flow pattern along the heated vertical plate with a sharp leading edge. This is not in line with Kulkarni *et al.*, who suggest agreement between this similarity solution and some numerical and experimental results in literature.

2. FLOW EQUATIONS

The Navier–Stokes equations for a laminar, two-dimensional, steady, incompressible flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+g\beta(T-T_\infty)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\nu}{Pr}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right). \tag{1}$$

The Boussinesq approximation has been applied. This means that the density ρ is considered constant everywhere, except in the temperature buoyancy term, where it is replaced by a linear dependence (constant coefficient of thermal expansion β) on the temperature difference $T-T_\infty$.

In the case where the characteristic number of the flow is very large (the Reynolds number in a forced convection flow, or the Grashof number $Gr = g\beta\Delta T x_0^3/\nu^2$ in a natural convection flow), boundary layers appear along fixed walls. In the boundary layer the Navier–Stokes description can be simplified to boundary-layer equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+g\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\nu}{Pr}\frac{\partial^2 T}{\partial y^2}. \tag{2}$$

We are searching for solutions of this system that describe the natural convection boundary-layer flow along a heated vertical plate with a sharp leading edge in a stratified environment

$$x = 0: \quad u \text{ and } T \text{ profile specified}$$
$$y = 0: \quad u = 0,\ T = T_w(x)$$
$$y \to \infty: \quad u = 0,\ T = T_\infty(x). \tag{3}$$

For special distributions $T_w(x)$ and $T_\infty(x)$ a similarity solution of equations (2) exists. Such a similarity solution depends only on one coordinate η, instead of the two independent x–y coordinates. Recently Semenov [2] has derived the differential equations for all possible similarity solutions. The temperature is rewritten as

$$T = (m + g(\eta))\Delta T \xi^n + T_c \tag{4}$$

with

$$\eta = 0: \quad g(\eta) = 1, \quad T_w = (m+1)\Delta T \xi^n + T_c$$
$$\eta \to \infty: \quad g(\eta) = 0, \quad T_\infty = m \Delta T \xi^n + T_c \tag{5}$$

where T_c is a constant. The transformed coordinates in this expression are

$$\xi = Mx + N \quad (\xi \geqslant 0)$$
$$\eta = \left(\frac{g\beta\Delta T}{v^2}|M|\right)^{1/4} (Mx+N)^{(n-1)/4} y. \tag{6}$$

A stream function is introduced as

$$\psi = \left(\frac{g\beta\Delta T v^2}{|M|^3}\right)^{1/4} \xi^{(n+3)/4} f(\eta) \tag{7}$$

which defines the u- and v-velocities as

$$u = \frac{\partial \psi}{\partial y} = \left(\frac{g\beta\Delta T}{|M|}\right)^{1/2} \xi^{(n+1)/2} f'$$
$$v = -\frac{\partial \psi}{\partial x} = \left(\frac{g\beta\Delta T v^2}{|M|^3}\right)^{1/4} M \xi^{(n-1)/4} \times \left(-\frac{n-1}{4}\eta f' - \frac{n+3}{4} f\right). \tag{8}$$

Substitution of these transformation expressions into equations (2) yields the following ordinary differential equations for f and g:

$$f''' + \operatorname{sgn}(M)\left[\frac{n+3}{4} f f'' - \frac{n+1}{2} f'^2\right] + g = 0$$
$$g'' + Pr\,\operatorname{sgn}(M)\left[\frac{n+3}{4} f g' - n(g+m) f'\right] = 0$$
$$\eta = 0: \quad f = f' = 0, \quad g = 1$$
$$\eta \to \infty: \quad f' = 0, \quad g = 0. \tag{9}$$

Special situations are $m = 0$, for the non-stratified environment, and $m = -1$, for the fixed wall temperature situation. The environment is stably stratified if $\mathrm{d}T_\infty/\mathrm{d}x > 0$, hence $mMn > 0$. The branch with $\operatorname{sgn}(M) = -1$ was discovered by Semenov.

Solutions of equations (9) have been determined by Ostrach [1] ($m = 0$, $\operatorname{sgn}(M) = 1$, $n = 0$), Sparrow and Gregg [3] ($m = 0$, $\operatorname{sgn}(M) = 1$, limited n-range), Cheesewright [4] and Yang *et al.* [5] ($m = -1$, $\operatorname{sgn}(M) = 1$, limited n-range) and Merkin [6] ($m = 0$, $\operatorname{sgn}(M) = 1$, whole n-range). No solutions are known for the whole n-range with $m = -1$ and for $\operatorname{sgn}(M) = -1$: these solutions are given here.

3. NUMERICAL METHOD

Two methods to solve the ordinary differential equations (9) are described: (i) the shooting method with explicit integration, and (ii) the direct method. The outer edge of the boundary layer is numerically taken at the finite distance η_∞. The region $0 \leqslant \eta \leqslant \eta_\infty$ is covered with the equidistantly spaced gridpoints η_j ($j = 0, 1, \ldots, J$).

Method (i) performs an explicit integration from the wall to the outer edge, finding the solution at $j+1$ from a Taylor expansion around j

$$\phi_{j+1} = \phi_j + \sum_{k=1}^{3} \frac{(\Delta\eta)^k}{k!} \phi_j^{(k)}, \quad \phi = f, g \tag{10}$$

where $\Delta\eta$ is the gridsize. The derivatives f''', g'' and g''' in this expression follow from equations (9). The remaining derivatives are obtained by Taylor expansions similar to equation (10). The integration can be started at the wall when the values $f(0)$, $f'(0)$, $f''(0)$, $g(0)$ and $g'(0)$ are known. The values $f(0)$, $f'(0)$ and $g(0)$ are given as boundary conditions, but $f''(0)$ and $g'(0)$ have to be guessed. Repeated integrations (shootings) are required to determine $f''(0)$ and $g'(0)$ such that the boundary conditions for f' and g at the outer edge η_∞ are satisfied. The iterative updating of $f'(0)$ and $g'(0)$ is performed with the Newton–Raphson method, requiring the numerical evaluation of

$$a_1 = \frac{\partial f'(\eta_\infty)}{\partial g'(0)}, \quad a_2 = \frac{\partial f'(\eta_\infty)}{\partial f''(0)},$$
$$a_3 = \frac{\partial g(\eta_\infty)}{\partial g'(0)}, \quad a_4 = \frac{\partial g(\eta_\infty)}{\partial f''(0)}. \tag{11}$$

$f''(0)$ and $g'(0)$ at the new iterative level s follow from the old level $s-1$ according to

$$f'(\eta_\infty) = f'^{s-1}(\eta_\infty) + (g'^s(0) - g'^{s-1}(0))a_1 + (f''^s(0) - f''^{s-1}(0))a_2$$
$$g(\eta_\infty) = g^{s-1}(\eta_\infty) + (g'^s(0) - g'^{s-1}(0))a_3 + (f''^s(0) - f''^{s-1}(0))a_4. \tag{12}$$

This Newton–Raphson process converges with a quadratic speed. The explicit integration (10) turned out to be very unstable; small deviations in the solution for $f''^s(0)$ and $g'^s(0)$ can lead to very large deviations in $f'^s(\eta_e)$ and $g^s(\eta_e)$.

The stability of method (ii) is much better. The equations are discretized according to

$$\frac{f_j - f_{j-1}}{\Delta\eta} - h_j = 0 \quad (j = 1,2,\ldots,J)$$

$$\frac{h_{j+1}-2h_j+h_{j-1}}{(\Delta\eta)^2} + \operatorname{sgn}(M)\left[\frac{n+3}{4} f_j \frac{h_{j+1}-h_{j-1}}{2\Delta\eta} - \frac{n+1}{2} h_j^2\right] + g_j = 0 \quad (j = 1,2,\ldots,J-1)$$

$$\frac{g_{j+1}-2g_j+g_{j-1}}{(\Delta\eta)^2} + Pr\operatorname{sgn}(M)\left[\frac{n+3}{4} f_j \frac{g_{j+1}-g_{j-1}}{2\Delta\eta} - n(g_j+m)h_j\right] = 0 \quad (j = 1,2,\ldots,J-1)$$

$$f_0 = 0, \quad h_0 = 0, \quad g_0 = 0, \quad h_J = 0, \quad g_J = 0. \tag{13}$$

A system of $3(J+1)$ non-linear algebraic equations results, which is solved with the Newton–Raphson method; at each iterative level the system is linearized and the resulting matrix equation is solved directly to update the solution. Discretization (13) yields a sparse Newton matrix (block tri-diagonal) in the matrix equation. The one-sided discretization for f' in equations (13) is only first-order accurate; the use of the second-order central discretization turned out to give an almost singular Newton matrix.

4. CALCULATED SIMILARITY SOLUTIONS

Solutions of equations (12) have been determined for air ($Pr = 0.72$).

Firstly the similarity solution for the situation with variable wall temperature and non-stratified environment ($m = 0$, $\operatorname{sgn}(M) = 1$) has been determined. Our results in Fig. 1 confirm the results of Merkin [6]; the solution becomes singular if $n \downarrow -0.999$, and no solution seems to exist for smaller values.

Secondly solutions have been determined for a fixed wall temperature and a stratified environment ($m = -1, \operatorname{sgn}(M) = 1$). Analogously to Merkin's analysis for $m = 0$ in the limit $n \to \infty$, the behaviour for $m = -1$ in the limit $|n| \to \infty$ can be found with the transformation

$$\eta = |n|^{-1/4}\bar{\eta}$$
$$f(\eta) = |n|^{-3/4}F(\bar{\eta})$$
$$g(\eta) = G(\bar{\eta}). \tag{14}$$

Substitution of equations (14) into equations (9) leads to

$$F''' + \operatorname{sgn}(M)\left[\frac{1}{4}\left(\operatorname{sgn}(n) + \frac{3}{|n|}\right)FF'' - \frac{1}{2}\left(\operatorname{sgn}(n) + \frac{1}{|n|}\right)F'^2\right] + G = 0$$

$$G'' + Pr\operatorname{sgn}(M)\left[\frac{1}{4}\left(\operatorname{sgn}(n) + \frac{3}{|n|}\right)FG' - \operatorname{sgn}(n)(G+m)F'\right] = 0$$

$$\bar{\eta} = 0: \quad F = F' = 0, G = 1$$
$$\bar{\eta} \to \infty: \quad F' = 0, G = 0. \tag{15}$$

This transformation gives the following relations for the wall-shear stress and wall-heat transfer

$$f''(0) = |n|^{-1/4}F''(0)$$
$$g'(0) = |n|^{1/4}G'(0). \tag{16}$$

It follows from equations (15) that F and G become independent of n in the limit $|n| \to \infty$. For $\operatorname{sgn}(M) = 1$ the wall-shear stress and wall-heat transfer are plotted in Fig. 2. Some velocity and temperature profiles are given in Fig. 3. It is seen that the whole n-range is free from singularities. As shown in detail in Fig. 4, a region with small backflow and temperature deficit is found in the outer part of the boundary layer in a stably stratified environment

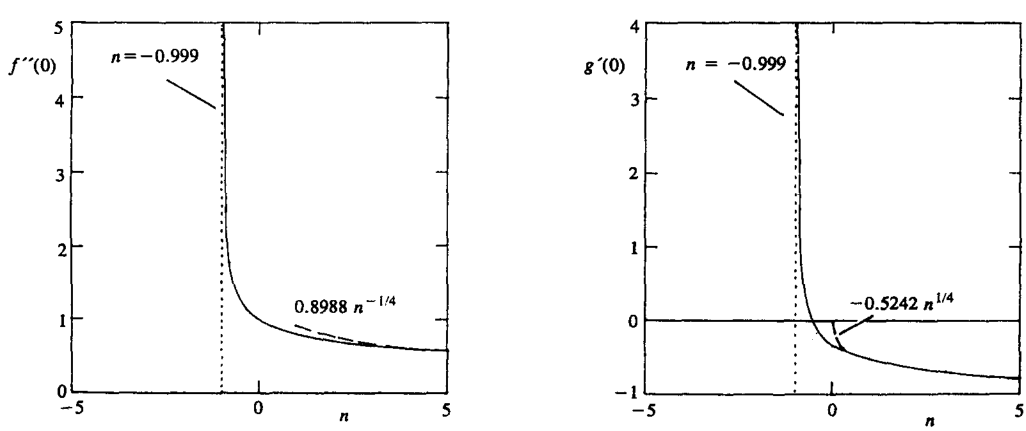

FIG. 1. Wall-shear stress (a) and wall-heat transfer (b) for variable wall and fixed environment temperature ($m = 0$, $\operatorname{sgn}(M) = 1$).

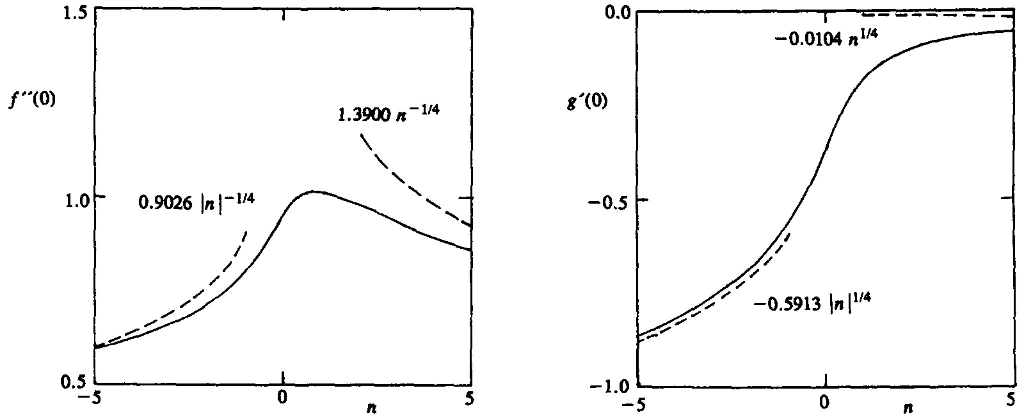

FIG. 2. Wall-shear stress (a) and wall-heat transfer (b) for fixed wall and variable environment temperature ($m = -1$, $\operatorname{sgn}(M) = 1$).

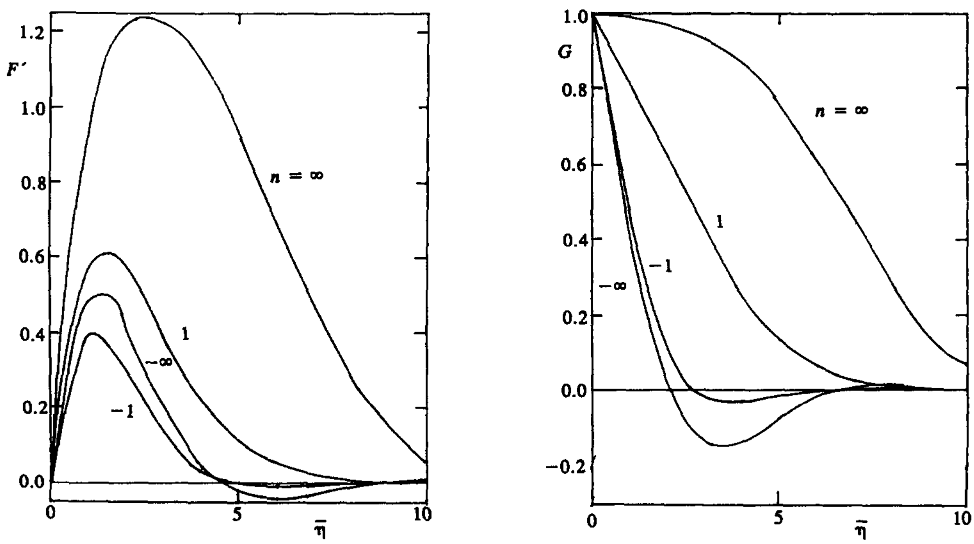

FIG. 3. Velocity (a) and temperature (b) profiles for fixed wall and variable environment temperature ($m = -1$, $\operatorname{sgn}(M) = 1$).

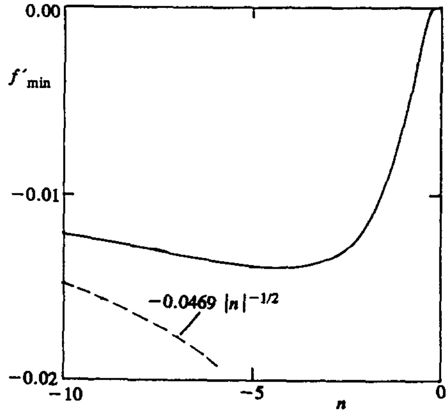

FIG. 4. Backflow in boundary layer for stably stratified environment ($m = -1$, $\operatorname{sgn}(M) = 1$).

($n < 0$). There is no backflow or temperature deficit in an unstably stratified environment ($n > 0$). The wall-shear stress and wall-heat transfer for $\operatorname{sgn}(M) = -1$ are given in Fig. 5. It is noted that in the limit $|n| \to \infty$, the solutions for $\operatorname{sgn}(M) = -1$ with $\operatorname{sgn}(n) = \pm 1$ is identical to the solution for $\operatorname{sgn}(M) = 1$ with $\operatorname{sgn}(n) = \mp 1$. Increasing n from $-\infty$ to 0 (unstable stratification) gives a zero wall-heat transfer with a temperature identical to 1 everywhere, except in a small region at the outer edge, where the temperature rapidly falls to the zero boundary condition. As illustrated in Fig. 6, the zero boundary condition for the velocity is satisfied in a small region at the outer edge as well. Although the negative n-branch for sgn (M) $= -1$ describes similarity solutions of the boundary-layer equations, they cannot

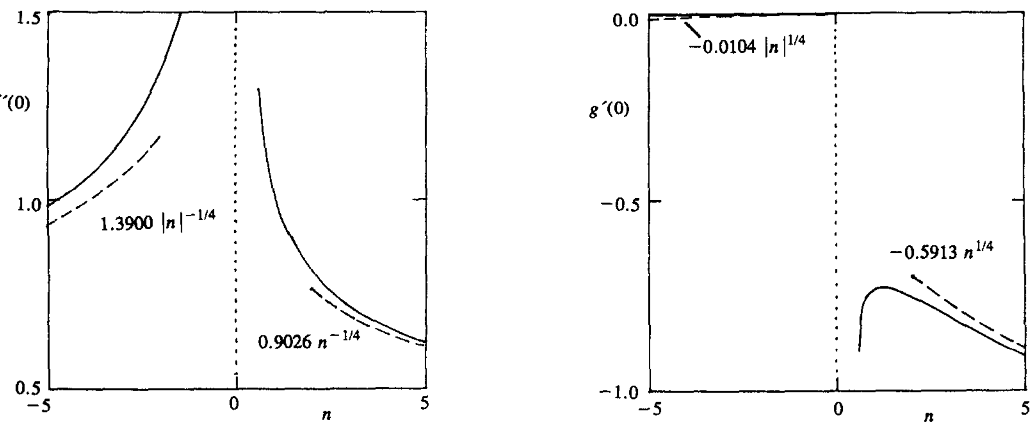

FIG. 5. Wall-shear stress (a) and wall-heat transfer (b) for fixed wall and variable environment temperature ($m = -1$, $\mathrm{sgn}(M) = -1$).

be part of the flow along the heated plate: the η-dependence of f' and g does not vanish if η is increased to infinity. This is required for the matching of the boundary-layer solution (inner solution) with the solution in the environment (outer solution) within the Navier–Stokes description. On the contrary, the velocity and temperature profiles in Fig. 7 show that this matching condition is satisfied for the solutions of the positive n-branch (stable stratification). As for the stable stratification with

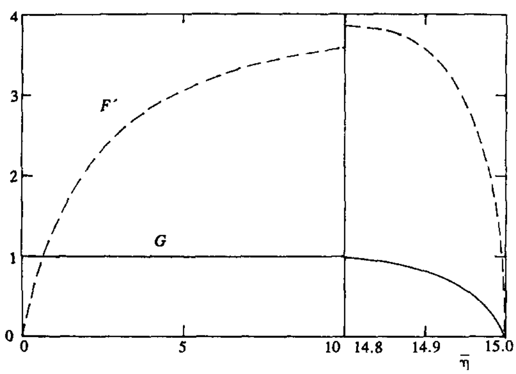

FIG. 6. Velocity and temperature profile for fixed wall temperature and unstably stratified environment ($m = -1$, $\mathrm{sgn}(M) = -1$, $n = -1$).

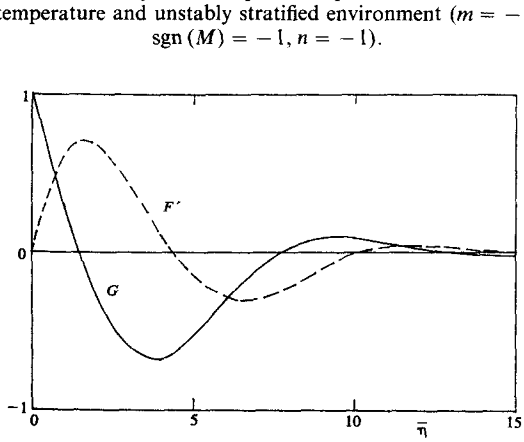

FIG. 7. Velocity and temperature profile for fixed wall temperature and stably stratified environment ($m = -1$, $\mathrm{sgn}(M) = -1$, $n = 1$).

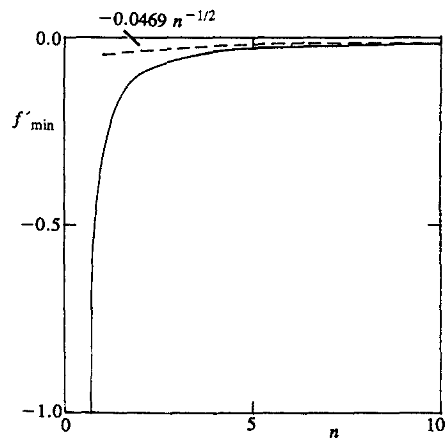

FIG. 8. Backflow in boundary layer for stably stratified environment ($m = -1$, $\mathrm{sgn}(M) = -1$).

$\mathrm{sgn}(M) = 1$, a region with backflow is found in the outer part of the boundary layer which is plotted in Fig. 8. Approaching $n \sim 0.6$ shows an enormous growth of $f''(0)$, $g'(0)$ and $f'_{\min}$, which seems to indicate the appearance of a singularity in the $\mathrm{sgn}(M) = -1$ branch.

5. MEANING OF THE SIMILARITY SOLUTIONS

For large Grashof numbers the Navier–Stokes solution along the vertical plate is described by boundary-layer equations. These boundary-layer equations do not hold in a small region, $O(Gr^{-1/2})$, at $x = 0$, where the full Navier–Stokes equations have to be used. If the temperature difference at $x = 0$ between wall and environment is nonzero, it has to be checked by solving the Navier–Stokes equations whether the solution in the $O(Gr^{-1/2})$ layer at $x = 0$ matches with Ostrach's similarity solution ($n = 0$). We verified this for a problem closely related to the vertical plate in an infinite stratified environment, namely for the heated vertical side in a square enclosure (for details see ref. [8]). If, however, the temperature difference between the wall and the environment at $x = 0$ is zero, the $n = 0$ similarity solution does not apply. Because a rising boundary layer at $x = 0$ requires that the wall temperature is not below the environment temperature, the zero temperature difference at $x = 0$ can only occur for an unstably stratified environment. Similarity solutions for this case indeed exist and are described by equations (9), with $\operatorname{sgn}(M) = 1$ and $n > 0$.

This implies that the other similarity solutions (the $\operatorname{sgn}(M) = 1$ class with $n < 0$ and the $\operatorname{sgn}(M) = -1$ class with $n > 0$) cannot be the boundary-layer solutions for small x. In order to check the meaning of these similarity solutions we have solved (non-similar) boundary-layer equations (2) for a stably stratified environment. The $n = 0$ similarity solution was used as a boundary condition at the initial $x = 0$. It was tried to solve the discretized boundary-layer equations in a single sweep, going from one x-station to the next larger x-station. However, the numerical iteration process failed to converge as soon as backflow occurred at an x-station. The reason for this failure seems to be clear: if backflow occurs, the boundary-layer equations locally change from a parabolic character to an elliptic character. This means that the single-sweep marching numerical solution technique has to be replaced by a repeated sweep procedure. Due to the elliptic character of the solution of the boundary-layer equations in a stably stratified environment, the solution at the last x-station has to be given as a boundary condition.

Hence, if the plate is semi-infinite the solution for $x \to \infty$ has to be known. We expect that the similarity class $\operatorname{sgn}(M) = 1$ with $n < 0$ forms that large x limit. The coefficient n describes how fast the environment temperature approaches the wall temperature for increasing x. The value of N in equations (6) is unimportant in the limit $x \to \infty$. The non-similar boundary-layer equations (2) have been solved for the stable stratification

$$\frac{T_\infty}{\Delta T} = 1 - \frac{1}{x/x_0 + 1} \quad 0 \leqslant x/x_0 \leqslant \infty. \tag{17}$$

At the leading edge of the vertical plate ($x = 0$) the similarity solution $\operatorname{sgn}(M) = 1$ with $n = 0$ is prescribed. At a large x-value the similarity solution $\operatorname{sgn}(M) = 1$ with $n = -1$ is prescribed. Repeated sweeps are made in the numerical procedure. The calculated wall-heat transfer (Nusselt number) is shown in Fig. 9. The Nusselt number is defined as

$$Nu = -\left[\frac{\partial\left(\dfrac{T - T_\infty(0)}{\Delta T}\right)}{\partial(y/x_0)}\right]_w. \tag{18}$$

Figure 9 shows that the non-similar solution smoothly matches both similarity limits

$$\lim x/x_0 \downarrow 0 \; Nu\,Gr^{1/4} = 0.3571\left(\frac{x}{x_0}\right)^{-1/4}$$

$$\lim x/x_0 \to \infty \; Nu\,Gr^{1/4} = 0.5592\left(\frac{x}{x_0} + 1\right)^{-3/2}. \tag{19}$$

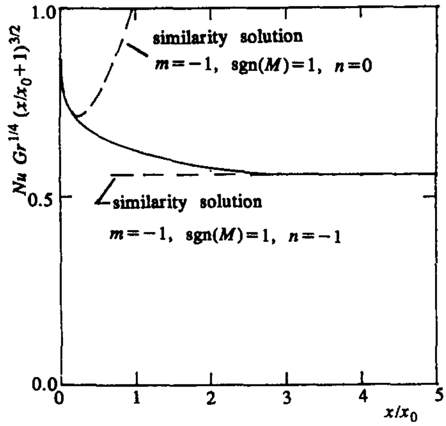

FIG. 9. Wall-heat transfer in non-similar boundary layer along semi-infinite vertical plate in stable stratification $T_\infty/\Delta T = 1 - 1/(x/x_0 + 1)$.

The environment temperature can cut the wall temperature at a finite distance x_0. In this case solving the (non-similar) boundary-layer equations requires a boundary condition at x_0. We expect that the non-similar solution matches the similarity class $\operatorname{sgn}(M) = -1$ with $n > 0$ in the limit $x \to x_0$. To check this, the non-similar boundary-layer equations (2) have been solved for a linear, stable stratification

$$\frac{T_\infty}{\Delta T} = \frac{x}{x_0} \quad 0 \leqslant x/x_0 \leqslant 1. \tag{20}$$

At $x = 0$ the similarity solution $\operatorname{sgn}(M) = 1$ with $n = 0$ is prescribed. At $x = x_0$ the similarity solution $\operatorname{sgn}(M) = -1$ with $n = 1$ is prescribed (using $M = -1/x_0$, $N = 1$). Actually only the u- and T-profiles have to be specified, and not the v-profile: $u = 0$ and $T = T_w$ at x_0. The same problem was also solved by Eichhorn [9] (with series expansions), by Chen and Eichhorn [10] (with the local non-similarity approximation method) and by Venkatachala and Nath [11] (with a finite difference numerical method). However, none of these authors discuss the need for a boundary condition at $x = x_0$ and the elliptic character of the boundary-layer equations. The calculated wall-heat transfer, velocity maximum and velocity minimum are depicted in Fig. 10. This figure shows that the solution smoothly matches the similarity solution for small x. In contrast with our expectation, the solution does *not* match the similarity solution for $x \to x_0$. In particular the wall-heat transfer behaves as

$$\lim x/x_0 \downarrow 0 \; Nu\,Gr^{1/4} = 0.3571\left(\frac{x}{x_0}\right)^{-1/4}. \tag{21}$$

But in the limit $x \to x_0$ the wall-heat transfer does not follow the similarity relation

$$\lim x/x_0 \uparrow 1 \; Nu\,Gr^{1/4} = 0.7313\left(1 - \frac{x}{x_0}\right). \tag{22}$$

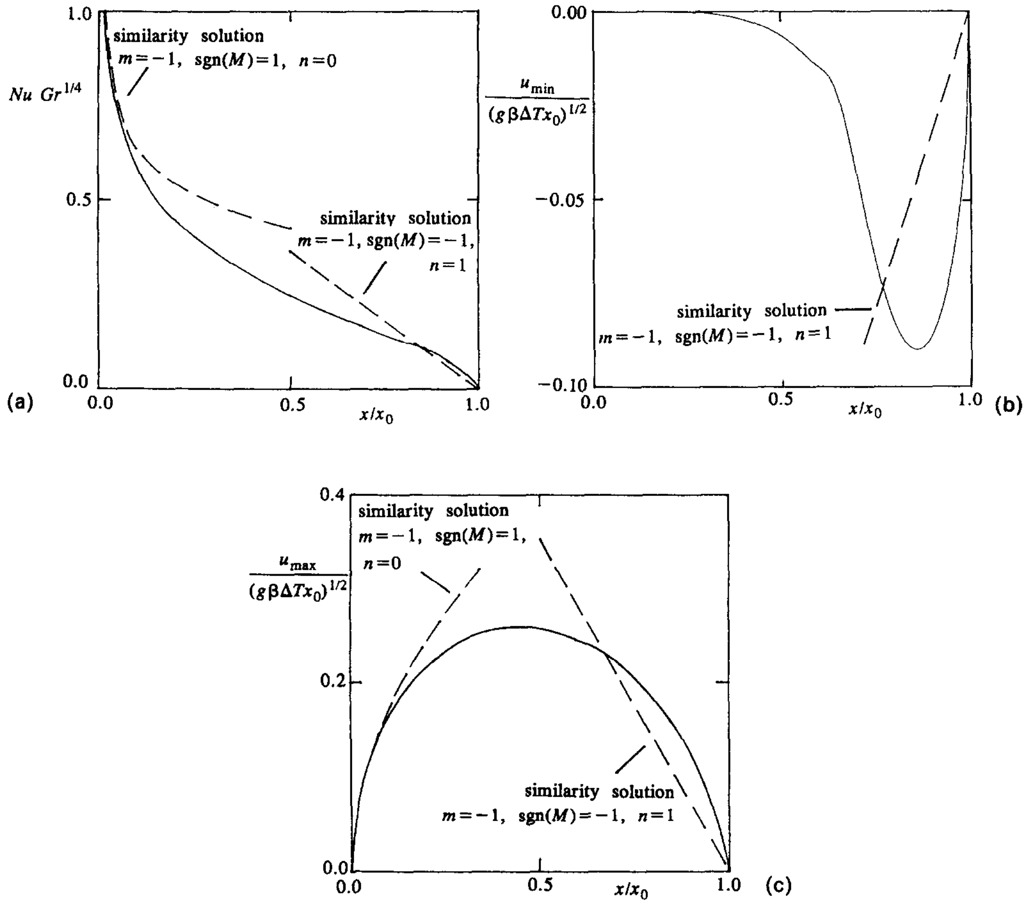

Fig. 10. Wall-heat transfer (a), velocity minimum (b) and velocity maximum (c) in non-similar boundary layer along finite vertical plate in stable, linear stratification ($T_\infty/\Delta T = x/x_0$).

6. CONCLUSION

Solving Semenov's differential equations for air, describing all possible similarity solutions of the natural convection boundary layer equations, shows that no singularity occurs in the positive M class for a fixed wall and variable environment temperature. Similarity solutions of the negative M class for an unstable stratification are not usable because the solutions do not smoothly match with the environment velocity and temperature.

Regions with backflow and temperature deficit are found in the similarity solutions for a stably stratified environment. The boundary-layer equations change from the parabolic to the elliptic type when regions with backflow occur, and a single sweep marching numerical technique to determine a non-similar solution has to be replaced by a multiple sweep technique. Besides the solution at the first x-station, also the solution at the last x-station has to be given as a boundary condition.

The similarity solution for a constant wall and environment temperature can be used to initiate the (non-similar) boundary-layer calculation for a heated plate with a sharp leading edge in a stably stratified environment. The similarity class with positive M and $n > 0$ gives the initial solution in the case where the stratification is unstable and the wall and environment temperature are equal at the leading edge. The similarity class with positive M and $n < 0$ gives the solution in a stable stratification for large x. On the contrary, if the environment temperature in a stable stratification becomes equal to the wall temperature at a finite distance x_0, the solution close to x_0 does not match the similarity solution of the negative M class with $n > 0$.

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ECOULEMENT A COUCHE LIMITE DE CONVECTION NATURELLE LAMINAIRE, LE LONG D'UNE PLAQUE CHAUDE VERTICALE, DANS UN ENVIRONNEMENT STRATIFIE

Résumé—On détermine numériquement toutes les solutions de similitude des équations de convection naturelle laminaire d'air à couche limite, pour une température de paroi fixée et d'environnement variable. On trouve que la classe à M positif n'a pas la singularité trouvée par Merkin pour une température de paroi variable et d'environnement fixé. Des solutions pour la classe à M négatif, pour une stratification instable, dépend de la position du bord extérieur et sont inutilisables. Les solutions de similitude pour une stratification stable montre des régions de retour. Néanmoins le calcul des solutions de non-similitudes des équations de couche-limite le long d'une plaque chaude verticale avec un bord d'attaque effilé demande que la solution soit connue à l'extrémité de la plaque. La classe à M positif fournit la solution pour une plaque semi-infinie. Si la température de l'environnement devient égale à celle de la paroi à une distance finie x_0, la solution non-similaire n'approche pas régulièrement la solution similaire de la classe à M négatif, près de x_0.

LAMINARE NATÜRLICHE GRENZSCHICHTSTRÖMUNG ENTLANG EINER BEHEIZTEN VERTIKALEN PLATTE IN GESCHICHTETER UMGEBUNG

Zusammenfassung—Die über Ähnlichkeitsbeziehungen aus den Grenzschichtgleichungen der laminaren natürlichen Konvektionsströmung gefundenen numerischen Lösungen für Luft werden bei konstanter Wand- und variabler Fluidtemperatur ermittelt. Es zeigt sich, daß die positive M-Klasse nicht die Singularität aufweist, welche von Merkin für die Strömung mit variabler Wand- und konstanter Fluidtemperatur gefunden wurde. Die Ergebnisse der negativen M-Klasse für eine instabile Schichtung hängen von der Position der äußeren Kante ab und sind deshalb unbrauchbar. Die Ähnlichkeitslösung für die stabil geschichtete Strömung zeigt Regionen mit Rückströmung. Bei einer beheizten vertikalen Platte mit einer scharfen Anströmungskante setzt deshalb die Berechnung von Lösungen, die sich nicht über die Ähnlichkeitsbeziehungen aus den Grenzschichtgleichungen ergeben, die Kenntnis der Lösung am Ende der Platte voraus. Die positive M-Klasse ergibt die Lösung für die halbunendliche Platte. Wenn die Fluidtemperatur in einem endlichen Abstand x_0 der Wandtemperatur entspricht, erreicht die Lösung, die nicht über die Ähnlichkeitsbeziehungen ermittelt wurde, nicht die Ähnlichkeitslösung der negativen M-Klasse nahe x_0.

ЛАМИНАРНОЕ ТЕЧЕНИЕ В ПОГРАНИЧНОМ СЛОЕ У НАГРЕТОЙ ВЕРТИКАЛЬНОЙ ПЛАСТИНЫ В УСЛОВИЯХ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ И СТРАТИФИКАЦИИ ОКРУЖАЮЩЕЙ СРЕДЫ

Аннотация—Численно получены все автомодельные решения уравнений, описывающих ламинарное течение воздуха в пограничном слое в условиях естественной конвекции при постоянной температуре стенки и переменной температуре окружающей среды. Найдено, что решение из положительного класса M не имеет сингулярности, как это было установлено Меркином для случая изменяющейся температуры стенок и постоянной температуры окружающей среды. Решения отрицательного класса M при неустойчивой стратификации зависят от положения внешней кромки и не имеют физического смысла. Из автомодельных решений для устойчивой стратификации следует существование областей с обратным течением. Поэтому для нахождения автомодельных решений уравнений пограничного слоя у нагретой вертикальной пластины с острой передней кромкой необходимо, чтобы было известно решение для края пластины. Положительный класс M содержит решение для полуограниченной пластины. При равенстве температур окружающей среды и стенок на конечном расстоянии x_0 неавтомодельное решение не стремится гладко к автомодельному решению отрицательного класса M вблизи x_0.